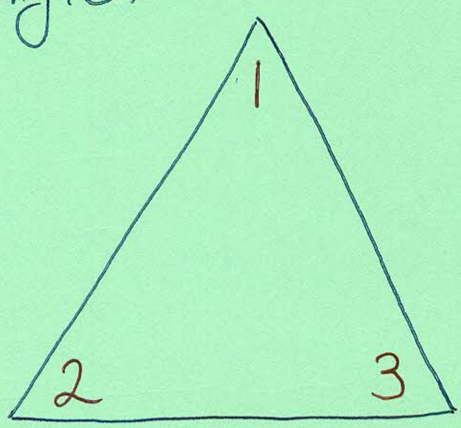
# Algebra and Geometry Unit

As our motivational problem, we will study the symmetries of a regular polygon, starting with an equilateral triangle.

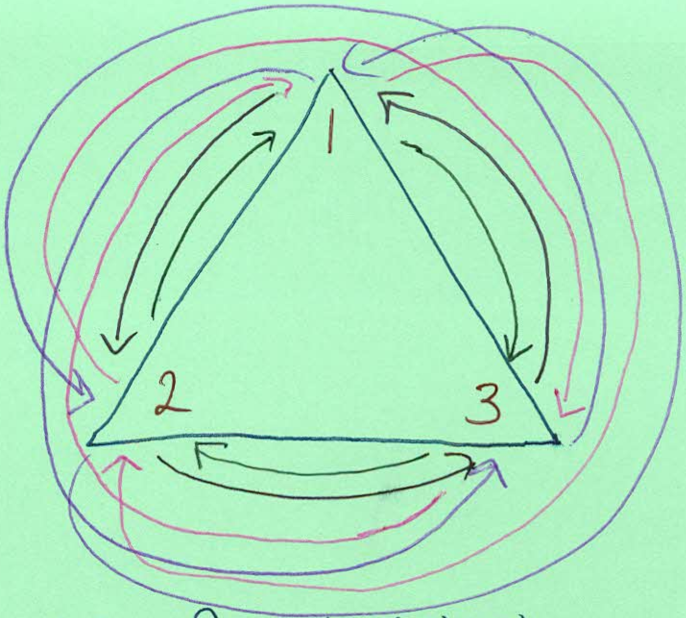
To do this, we begin by numbering the corners of the triangle:




Now, let's find all the ways we can move the triangle without changing the shape. There are two basic moves we can do:

- rotations
- reflections

## Rotations




1) Do Nothing : $e = R_0 = (1)(2)(3)$

2) Counterclockwise rotation by $\frac{2\pi}{3}$ : $(1\ 2\ 3) = R_{\frac{2\pi}{3}}$

3) Counterclockwise rotation by $\frac{4\pi}{3}$ : $(1\ 3\ 2) = R_{\frac{4\pi}{3}}$

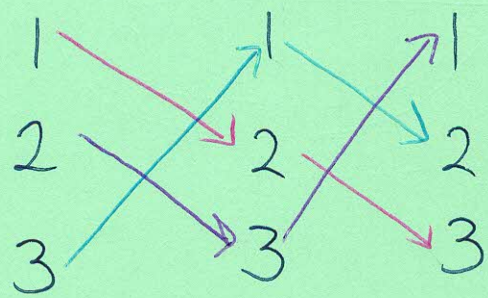
4) Clockwise rotation by $\frac{2\pi}{3}$ : $(1\ 3\ 2) = R_{-\frac{2\pi}{3}}$

5) Clockwise rotation by $\frac{4\pi}{3}$ : $(1\ 2\ 3) = R_{-\frac{4\pi}{3}}$

Observe that $R_{\frac{2\pi}{3}} = R_{-\frac{4\pi}{3}}$ & $R_{\frac{4\pi}{3}} = R_{-\frac{2\pi}{3}}$, so we only need to worry about counterclockwise rotations. Of course, rotating by $\frac{2\pi}{3}$, then $\frac{2\pi}{3}$ is the same as rotating by $\frac{4\pi}{3}$, so $R_{\frac{4\pi}{3}} = \left(R_{\frac{2\pi}{3}}\right)^2$. Let's look at how to multiply the cycle notation:

We know that $(1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2)$.

We see this by

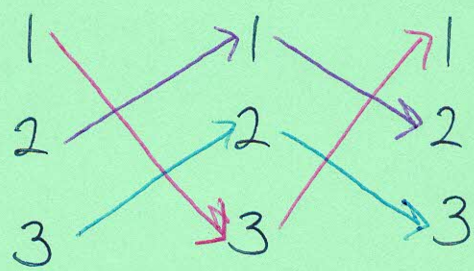


$$(1\,2\,3) \circ (1\,2\,3) = (1\ 3\ 2)$$

We should also expect that

$$\left(R_{\frac{2\pi}{3}}\right)^3 = R_{\frac{4\pi}{3}} \circ R_{\frac{2\pi}{3}} = R_{2\pi} = R_0$$

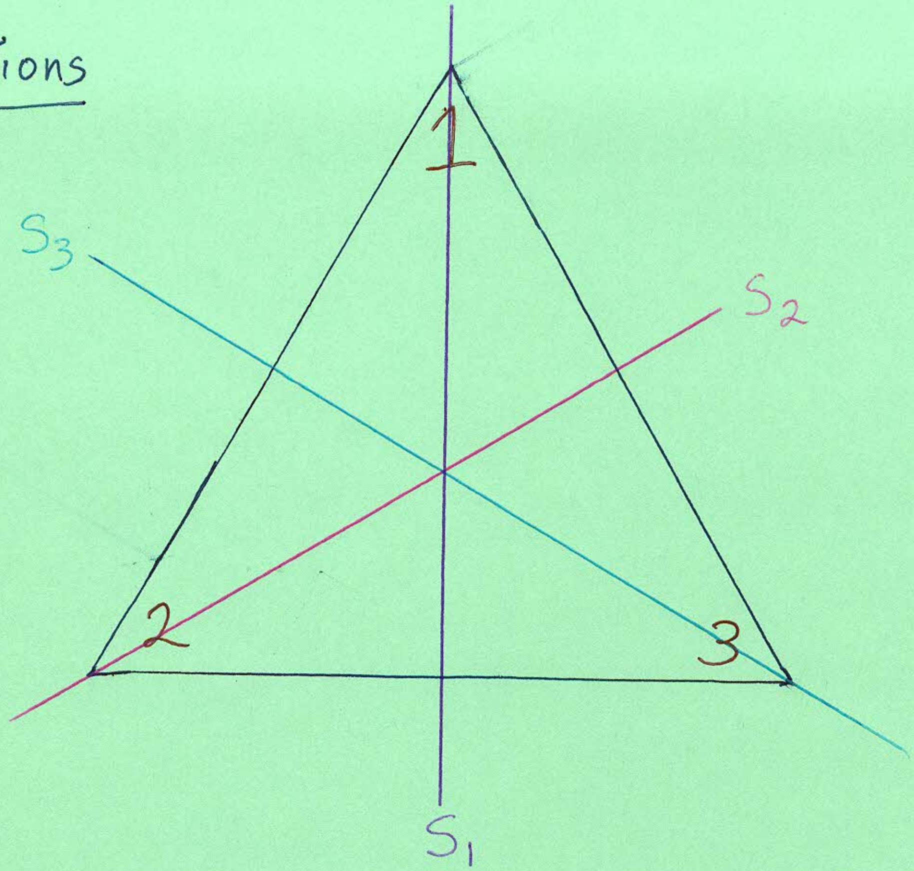
Let's see it with cycles:



$$(1\ 3\ 2) \cdot (1\ 2\ 3) = (1)(2)(3) = e$$

[*Normally we leave off any singletons, and write "e" if everything is a singleton.*]

# Reflections




There are three lines we can reflect across while maintaining the shape, let's call these reflections $s_1$, $s_2$, and $s_3$. In cycle notation:

$$S_1 = (1)(2\,3) = (2\,3)$$

$$S_2 = (2)(1\,3) = (1\,3)$$

$$S_3 = (3)(1\,2) = (1\,2)$$

Of course, reflecting across a line, then across it again should bring us back to the starting point, i.e.,

$$(S_1)^2 = e, \ (S_2)^2 = e, \ \& \ (S_3)^2 = e$$